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APPLICATIONS OF TIED IMPLICATIONS TO APPROXIMATE REASONING AND FUZZY CONTROL

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Abstract

The logic of tied implications deals with implications $\Rightarrow: P \times L \to L$ (on two lattices $(P, \leq_P, 1_P)$ and $(L, \leq_L)$), tied by an integral commutative ordered monoid operation $\otimes$ on P, in the sense of the following identity: $((x \otimes y) \Rightarrow c) = (x \Rightarrow (y \Rightarrow c))$. We demonstrate the usefulness of tied implications through some applications. We use the connectives of a tied algebra to interpret Generalized Modus Ponens (**GMP**) inference schemata, in the vein of both the Compositional Rule of Inference (**CRI**) of Zadeh, and the Consequent Dilation Rule (**CDR**), due to Magrez and Smets and developed by Morsi and Fahmy. We show that a multiple-rule, generalized modus ponens inference scheme is equivalent, as far as **CRI** or **CDR** are concerned, to a scheme that satisfies the "basic requirement for fuzzy reasoning", proposed by Fukami, Lehmke, Perfilieva, Tian and Turksen. We end by investigating the principles of fuzzy control in general with interpretations based on a particular case of **CRI**, called Generalized Conjunctive Rule (**GCR**), due to Hájek. We show that the basic requirement for fuzzy reasoning is satisfied by **GCR** in all single rule inference schemata, using the connectives of a tied algebra, and we indicate a special type of multiple-rule schemata in which this requirement is satisfied.

Keywords

Fuzzy logic; Generalized modus ponens; Fuzzy control; Nonclassical logics; Tied implication; Prelinearity

1. Introduction

By an implication on two lattices P and L, we mean a function $\Rightarrow: P \times L \to L$ with mixed monotonicity properties, and with the top element 1_P of P as a left identity element. This $\Rightarrow$ is said to be *tied* if there is an integral commutative ordered monoid operation $\otimes$ on $(P, 1_P)$, called here an *object-conjunction* (also called the *tying-conjunction*), such that the following identity holds: $((x \otimes y) \Rightarrow c) = (x \Rightarrow (y \Rightarrow c))$. This property extends to multiple-valued logic the equivalence, in classical logic, of the following two statements: "If (X and Y) then C", and "If X then (if Y then C)". It holds for several types

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of implications used in fuzzy logic. Our study of tied implications is founded in the well-established domain of residuated lattices, whereby implications and conjunctions are related by residuation (= adjointness).

Section 2.1 offers needed background on the notion of adjointness. We follow Morsi [13] in the simultaneous use of two lattices in one adjointness algebra. However, in the special case of a residuated algebra (Section 2.2), only one lattice can be used, because the conjunction there has a two-sided identity element.

In Section 2.3, we quote the Morsi - Abdel-Hamid definition of tied algebras [1,13]. We request tying-conjunctions to be commutative integral ordered monoid operations. We require them to be supremum-preserving, in order to have residuated implications. Those structures admit an abundance of useful inequalities [15], which we compile in Theorem 3. Yet they encompass a wide variety of examples [15], among which the residuated algebras are one type. In Section 2.4 we quote from [15,16] the definition and principal properties of *prelinear tied algebras* (over pairs of lattices), defined in analogy with the prelinear structures of Esteva, Godo, Hájek and Höhle.

We demonstrate the usefulness of tied implications through some applications. In Section 3, we use the connectives of a tied adjointness algebra to interpret Generalized Modus Ponens (**GMP**) inference schemata, in the vein of both the Compositional Rule of Inference (**CRI**) of Zadeh (Section 3.2), and the Consequent Dilation Rule (**CDR**), originally due to Magrez and Smets [10], and developed by Morsi and Fahmy [14] (Section 3.3). We show that a multiple-rule, generalized modus ponens inference scheme is equivalent, as far as **CRI** or **CDR** are concerned, to a system that satisfies the "basic requirement for fuzzy reasoning", proposed by Fukami et al. [7], Turksen and Tian [21], and Perfilieva and Lehmke [19,20]. In Section 4, we look into some general principles of fuzzy control with interpretations based on a particular case of the **CRI**, called Generalized Conjunctive Rule (**GCR** for short), due to Hájek [8]. We show that the "basic requirement for fuzzy reasoning" is satisfied by **GCR** in all single rule inference schemes, when the connectives of a tied algebra are properly employed. But, it is satisfied in a special type of multiple-rule inference schemes only. We indicate clearly where prelinearity is needed in some (but not all) of our proofs.

Our conclusions are given summarily in Section 5.

2. Basics

2.1. Implications and their adjoints

Throughout this article, $(P, \leq_P)$ and $(L, \leq_L)$ denote two independently chosen complete lattices, usually denoted simply by P and L, respectively. The top element of the lattice $(P, \leq_P)$ is denoted by 1_P. An *implication*[3] (Morsi [13]) $\Rightarrow$ on P, L is a function : $P \times L \to L$ that satisfies the following five conditions:

I1: $\Rightarrow$ is antitone in the left argument.

I2: $\Rightarrow$ is isotone in the right argument.

I3: $\Rightarrow$ has 1_P as a left identity element.

[3]This two-lattices approach to logical connectives, was started algebraicly by Morsi in [13]. It was then adopteded by Morsi, Lotfallah and El-Zekey [15], and by Morsi and Roshdy [17]. Then it was formulated syntactically, within the first order logic of tied implications, by Morsi, Lotfallah and El-Zekey in [16].

I4: $\Rightarrow$ has an adjoint $\supset$ in the left argument; that is, $\supset: L \times L \to P$ is a function that satisfies the following equivalences for all $x \in P,\ b, c \in L$:

$$\textbf{Adjointness 1}: \quad x \leq_P b \supset c \quad \text{if and only if} \quad b \leq_L x \Rightarrow c. \tag{1}$$

I5: $\Rightarrow$ has an adjoint & in the right argument; that is, $\& : P \times L \to L$ is a function that satisfies the following equivalences for all $x \in P,\ b, c \in L$:

$$\textbf{Adjointness 2}: \quad x\&b \leq_L c \quad \text{if and only if} \quad b \leq_L x \Rightarrow c. \tag{2}$$

The presence of those adjoints $\supset$ and & helps to generate universal inequalities in the ensuing calculus, and so facilitates the solution therein of inequalities that may arise during various applications.

The two adjoints $\supset$ and & of $\Rightarrow$ exist if and only if the implication $\Rightarrow$ satisfies:

$$\left(\sup_j x_j \Rightarrow \inf_m c_m\right) = \inf_{j,m} (x_j \Rightarrow c_m), \tag{3}$$

for all indexed families $\{x_j\}$ in P, and $\{c_m\}$ in L. They are then uniquely given by

$$b \supset c = \sup \{x \in P : \quad b \leq_L x \Rightarrow c\}, \quad b, c \in L. \tag{4}$$

$$x\&b = \inf \{c \in L : \quad b \leq_L x \Rightarrow c\}, \quad x \in P,\ b \in L. \tag{5}$$

Thus, we can rephrase the definition of an implication $\Rightarrow: P \times L \to L$ as a function which satisfies (3), and has 1_P as a left identity element.

The function $\supset: L \times L \to P$ is called a *comparator*. A comparator satisfies a condition analogous to (3), and determines the partial order on L in manner of the following *comparator property*:

$$b \leq_L c \quad \text{if and only if} \quad b \supset c = 1_P, \quad \forall b, c \in L; \tag{6}$$

which confers on comparators their nomenclature.

We call the function $\& : P \times L \to L$ a *metalogical conjunction*. This is due to its role in an interpretation of *generalized modus ponens* (Section 3.2), whereby it conjoins premises (for instance, an observation with a rule), as a step toward producing conclusions, see also [12,15]. A metalogical conjunction has 1_P as a left identity element and preserves, in each argument, all suprema that may exist in P or L, that is

$$\left(\sup_j x_j \& \sup_m c_m\right) = \sup_{j,m} (x_j \& c_m), \tag{7}$$

for all indexed families $\{x_j\}$ in P, and $\{c_m\}$ in L.

It is directly seen that each comparator $\supset$, or each metalogical conjunction &, is the adjoint of a unique implication $\Rightarrow$, obtained through either one of the following two equations: For all $x \in P,\ c \in L$:

$$x \Rightarrow c = \sup \{b \in L : \quad x \leq_P b \supset c\}, \tag{8}$$

$$x \Rightarrow c = \sup \{b \in L : \quad x\&b \leq_L c\}. \tag{9}$$

When adjointness is applied to the following three trivial identities $x \Rightarrow c = x \Rightarrow c$, $x\&b = x\&b$ (in L) and $b \supset c = b \supset c$ (in P), we obtain, for all $x \in P$ and all $b, c \in L$, the following six basic inequalities in P and L:

$$x \leq_P \quad (x \Rightarrow c) \supset c, \tag{10}$$

$$x\&(x \Rightarrow c) \leq_L \quad c, \tag{11}$$

$$b \leq_L \quad x \Rightarrow (x\&b), \tag{12}$$

$$x \leq_P \quad b \supset (x\&b), \tag{13}$$

$$b \leq_L \quad (b \supset c) \Rightarrow c, \tag{14}$$

$$(b \supset c)\&b \leq_L \quad c. \tag{15}$$

Definition 1 *The mathematical system* $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset)$*, described above, is called an* adjointness algebra*. The class of all adjointness algebras is denoted by* $|\mathbb{ADJA}|$*.*

We need the following universal inequalities in adjointness algebras:

$$\inf_j (b_j \supset c_j) \leq_P (\inf_j b_j \supset \inf_j c_j) \tag{16}$$

$$\inf_j (b_j \supset c_j) \leq_P (\sup_j b_j \supset \sup_j c_j) \tag{17}$$

for all indexed families $\{b_j\}$ and $\{c_j\}$ in P, such that the suprema and infima in the right-hand sides exist.

An implication $\Rightarrow$ is said to be *faithful*[4] if it satisfies:

$$(\forall x, y \in P) \;\; (\text{if } x \neq y, \text{ then } (\exists c \in L): \; x \Rightarrow c \neq y \Rightarrow c). \tag{18}$$

It is said to satisfy the *exchange axiom* if it satisfies:

$$(\forall x, y \in P)(\forall c \in L) \;\; (x \Rightarrow (y \Rightarrow c) = y \Rightarrow (x \Rightarrow c)). \tag{19}$$

2.2. Residuated implications

We say that $\Rightarrow, \supset, \&$ are connectives *on* P whenever $(L, \leq_L) = (P, \leq_P)$. If, in this case, $\&$ is commutative and associative, we call it an object-level-conjunction, or shortly, an *object-conjunction*. This is drawn from its role in generalized modus ponens, whereby it conjoins subformulae within one premise, see [15, Section 4]. Whenever & is commutative, $\Rightarrow$ has to coincide with $\supset$; becoming simultaneously an implication and a comparator. We usually denote an object-conjunction by $\otimes$. We denote its adjoint $\Rightarrow = \supset$ by $\rightarrow$. A vast literature is devoted to the mathematical system $(P, \leq, 1, \otimes, \rightarrow)$, called a *residuated algebra*, whereby $\rightarrow$ is called the *R-implication*, *residuum*, or *residuated implication*, of $\otimes$[5]. The class of all residuated algebras is denoted by $|\mathbb{RA}|$.

The algebraic properties of residuated algebras $(P, \leq, 1, \otimes, \rightarrow)$ are now well known. In particular, we shall have recourse to:

Strong residuation : $(x \rightarrow (y \rightarrow z)) = (y \rightarrow (x \rightarrow z)) = (x \otimes y \rightarrow z)$,

Residuation : $x \otimes y \leq_P z \quad iff \quad y \leq_P x \rightarrow z \quad iff \quad x \leq_P y \rightarrow z.$

[4] It is said to *distinguish left arguments* in [1,15,17]

[5] Object-conjunctions on the unit interval $[0, 1]$ of real numbers are all the supremum-preserving triangular norms (t-norms).

2.3. Tied implications

A binary operation $\star$ (on P) *ties* an implication $\Rightarrow$ (on P, L) if the following identity holds:

$$(\forall x, y \in P)\,(\forall c \in L) \quad (((x \star y) \Rightarrow c) = (x \Rightarrow (y \Rightarrow c)))\,, \tag{20}$$

and we say that $\Rightarrow$ is *tied* [1]. Every tied implication $\Rightarrow$ on P, L has a greatest binary operation $\otimes_{\Rightarrow}$ on P that ties it, and $\otimes_{\Rightarrow}$ is associative and isotone in each argument. It becomes commutative if and only if $\Rightarrow$ satisfies the exchange axiom [1]. If $\Rightarrow$ is tied and faithful, then it is tied uniquely by $\otimes_{\Rightarrow}$. In this case, $\otimes_{\Rightarrow}$ preserves arbitrary suprema, and 1_P becomes a two-sided identity element for $\otimes_{\Rightarrow}$; that is, $\otimes_{\Rightarrow}$ becomes a (possibly noncommutative) object-conjunction [1]. For this reason, as well as other reasons expounded in [15, Remark 3.1], we shall henceforth reserve the term *tied implication* only for implications tied by (commutative) object-conjunctions.

Definition 2 *The ensuing mathematical system $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset, \otimes, \rightarrow)$ is called a tied adjointness algebra [13,15], or shortly, a* tied algebra*. The class of all tied algebras is denoted by $|\mathbb{TA}|$. A* tied chain *(a linear* tied algebra*) is one in which both lattices are chains.*

It is well known that $\otimes$ ties its residuum $\rightarrow$. Hence, every residuated algebra $(P, \leq_P, 1_P, \otimes, \rightarrow)$ is automatically a tied algebra of the type $(P, \leq_P, 1_P, P, \leq_P, \rightarrow, \otimes, \rightarrow, \otimes, \rightarrow)$.

Tied algebras provide a particularly rich framework, through which all useful properties of residuated algebras can extend over a much wider scope of logical connectives already in use in fuzzy logic, see the examples of tied algebras in [15]. In the meantime, we manage to retain all their desirable traits, but with the roles of $\otimes$ and $\rightarrow$ being distributed among the five connectives of a tied algebra. This is attested by the thirty schemata of formulae, in the next theorem, which ensue from the tiedness of $\Rightarrow$.

Theorem 3 *[15] The following identities and inequalities hold in a tied algebra $(P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset, \otimes, \rightarrow)$: $\forall x, y, z \in P$, $\forall a, b, c, d \in L$:*

$$\otimes \ ties \ \Rightarrow: ((x \otimes y) \Rightarrow c) = (x \Rightarrow (y \Rightarrow c))\,, \tag{21}$$

$$\otimes \ ties \ \& : ((x \otimes y)\,\&c) = (x\&\,(y\&c))\,, \tag{22}$$

$$Strong\ adjointness : (b \supset (x \Rightarrow c)) = (x \rightarrow (b \supset c)) = (x\&b \supset c)\,, \tag{23}$$

$$Exchange\ axiom\ for\ \Rightarrow: (x \Rightarrow (y \Rightarrow c)) = (y \Rightarrow (x \Rightarrow c))\,, \tag{24}$$

$$Exchange\ axiom\ for\ \& : (x\&\,(y\&c)) = (y\&\,(x\&c))\,, \tag{25}$$

$$((x \rightarrow y) \otimes (c \supset d)) \leq_P ((y \Rightarrow c) \supset (x \Rightarrow d))\,, \tag{26}$$

$$((x \rightarrow y) \otimes (c \supset d)) \leq_P ((x\&c) \supset (y\&d))\,, \tag{27}$$

$$((a \supset b) \otimes (c \supset d)) \leq_P ((b \supset c) \rightarrow (a \supset d))\,, \tag{28}$$

$$\supset \ is \ \otimes-transitive : ((a \supset b) \otimes (b \supset d)) \leq_P (a \supset d)\,, \tag{29}$$

$$Prefixing\ with\ \Rightarrow: b \supset c \leq_P ((x \Rightarrow b) \supset (x \Rightarrow c))\,, \tag{30}$$

$$Prefixing\ with\ \&: b \supset c \leq_P ((x\&b) \supset (x\&c)), \tag{31}$$
$$Prefixing\ with\ \supset: b \supset c \leq_P ((a \supset b) \to (a \supset c)), \tag{32}$$
$$Suffixing\ with\ \Rightarrow: x \to y \leq_P ((y \Rightarrow c) \supset (x \Rightarrow c)), \tag{33}$$
$$Suffixing\ with\ \&: x \to y \leq_P ((x\&b) \supset (y\&b)), \tag{34}$$
$$Suffixing\ with\ \supset: b \supset c \leq_P ((c \supset d) \to (b \supset d)), \tag{35}$$
$$Balance\ 1: (((b \supset a) \Rightarrow d) \supset ((c \supset a) \Rightarrow d)) \geq_P (b \supset c), \tag{36}$$
$$Balance\ 2: (((a \supset b)\&d) \supset ((a \supset c)\&d)) \geq_P (b \supset c), \tag{37}$$
$$Balance\ 3: (((x \Rightarrow (y \Rightarrow c)) \supset c) \Rightarrow d) \leq_L (x \Rightarrow (y \Rightarrow d)), \tag{38}$$
$$Balance\ 4: ((c \supset (x\&(y\&c)))\&d) \geq_L (x\&(y\&d)), \tag{39}$$
$$x\&(y \Rightarrow c) \leq_L (y \Rightarrow (x\&c)), \tag{40}$$
$$(y \to x)\&c \leq_L (y \Rightarrow (x\&c)), \tag{41}$$
$$y\&(x \Rightarrow c) \leq_L ((y \to x) \Rightarrow c), \tag{42}$$
$$(b \supset c)\&d \leq_L ((d \supset b) \Rightarrow c), \tag{43}$$
$$x \otimes (b \supset c) \leq_P ((x \Rightarrow b) \supset c), \tag{44}$$
$$x \otimes (c \supset b) \leq_P (c \supset (x\&b)), \tag{45}$$
$$x \otimes y \leq_P ((x \Rightarrow (y \Rightarrow c)) \supset c), \tag{46}$$
$$x \otimes y \leq_P (c \supset (x\&(y\&c))), \tag{47}$$
$$y \Rightarrow c \leq_L ((x \otimes y) \Rightarrow (x\&c)), \tag{48}$$
$$y \Rightarrow c \leq_L ((x \to y) \Rightarrow (x \Rightarrow c)), \tag{49}$$
$$x \Rightarrow b \leq_L ((b \supset c) \Rightarrow (x \Rightarrow c)). \tag{50}$$

2.4. Prelinear tied algebras

Definition 4 *(Morsi, Lotfallah and El-Zekey [15]) A* prelinear tied algebra *is a tied algebra* $\Lambda = (L, \leq_L, P, \leq_P, 1, \Rightarrow, \&, \supset, \otimes, \to, \wedge_P, \vee_P, \wedge_L, \vee_L)$ *that satisfies the following two prelinearity equations for* $\to$ *and* $\supset$*:*

$$\forall a, c \in P: \quad (a \to c) \vee_P (c \to a) = 1_P, \tag{51}$$
$$\forall x, y \in L: \quad (x \supset y) \vee_P (y \supset x) = 1_P. \tag{52}$$

We denote the class of all prelinear tied adjointness algebras by $|\mathbb{L}\text{-}\mathbb{TA}|$**.**

The terminology prelinear is due to Hájek (1998) in the setting of residuated algebras. It is justified by the fact that (51) and (52) hold trivially whenever P, L are linearly ordered (i.e., chains), due to the comparator axiom (6). These identities are equivalent in a tied algebra to the following inequalities [6,8,15]:

$$\forall z \in P, \forall b, c \in L: \quad ((b \supset c) \to z) \leq_P \quad (((c \supset b) \to z) \to z), \tag{53}$$
$$\forall x, y, z \in P: \quad ((x \to y) \to z) \leq_P \quad (((y \to x) \to z) \to z). \tag{54}$$

Prelinearity enters into many proofs through the two inferences of the next proposition.

Proposition 5 (Deduction by Cases) *[16]*

$$\textit{If } \{(\alpha \supset \beta) \leq_P \chi \textit{ and } (\beta \supset \alpha) \leq_P \chi\} \qquad \textit{then } \chi = 1_P, \tag{55}$$
$$\textit{If } \{(\xi \to \tau) \leq_P \chi \textit{ and } (\tau \to \xi) \leq_P \chi\} \qquad \textit{then } \chi = 1_P. \tag{56}$$

We need the following obvious identities and inequalities in $|\mathbb{TA}|$:

Theorem 6

$$\begin{aligned}
(\alpha \supset \beta) &= (\alpha \supset \alpha \wedge_L \beta), & (57)\\
(\alpha \supset \beta) &= (\alpha \vee_L \beta \supset \beta), & (58)\\
(\xi \to \tau) &= (\xi \to \xi \wedge_P \tau), & (59)\\
(\xi \to \tau) &= (\xi \vee_P \tau \to \tau), & (60)\\
(\alpha \supset \beta) &\leq_P (\alpha \wedge_L \gamma \supset \beta \wedge_L \gamma), & (61)\\
(\alpha \supset \beta) &\leq_P (\alpha \vee_L \gamma \supset \beta \vee_L \gamma). & (62)
\end{aligned}$$

The two lattices P and L, underlying a prelinear tied algebra, have to be distributive [16]. We next compile principal features of prelinear tied algebras.

Theorem 7 (Meet and Join) *[15,16]*

$$\begin{aligned}
\alpha \vee_L \beta &= ((\beta \supset \alpha) \Rightarrow \alpha) \wedge_L ((\alpha \supset \beta) \Rightarrow \beta), & (63)\\
\xi \vee_P \tau &= ((\tau \to \xi) \to \xi) \wedge_P ((\xi \to \tau) \to \tau), & (64)\\
\alpha \wedge_L \beta &= ((\beta \supset \alpha) \& \beta) \vee_L ((\alpha \supset \beta) \& \alpha), & (65)\\
\xi \wedge_P \tau &= (\tau \otimes (\tau \to \xi)) \vee_P (\xi \otimes (\xi \to \tau)). & (66)
\end{aligned}$$

We use the abbreviation τ^n to denote $\tau \otimes ... \otimes \tau$ (n copies of τ).

Theorem 8 *[6,8,16]*

$$\begin{aligned}
\zeta \otimes \tau &\leq_P \zeta^2 \vee_P \tau^2, & (67)\\
(\zeta \vee_P \tau) \otimes (\zeta \vee_P \tau) &\leq_P \zeta^2 \vee_P \tau^2, & (68)\\
(\alpha \supset \beta)^n \vee_P (\beta \supset \alpha)^n &= 1_P, & (69)\\
(\zeta \to \tau)^n \vee_P (\tau \to \zeta)^n &= 1_P \quad \textit{[8]}. & (70)
\end{aligned}$$

Theorem 9 (Lattice Homomorphisms) *[16]*

$$(\xi \Rightarrow \beta \wedge_L \gamma) = (\xi \Rightarrow \beta) \wedge_L (\xi \Rightarrow \gamma), \tag{71}$$
$$(\xi \vee_P \tau \Rightarrow \gamma) = (\xi \Rightarrow \gamma) \wedge_L (\tau \Rightarrow \gamma), \tag{72}$$
$$(\xi \Rightarrow \beta \vee_L \gamma) = (\xi \Rightarrow \beta) \vee_L (\xi \Rightarrow \gamma), \tag{73}$$
$$(\xi \wedge_P \tau \Rightarrow \gamma) = (\xi \Rightarrow \gamma) \vee_L (\tau \Rightarrow \gamma), \tag{74}$$
$$(\alpha \supset \beta \wedge_L \gamma) = (\alpha \supset \beta) \wedge_P (\alpha \supset \gamma), \tag{75}$$
$$(\alpha \vee_L \beta \supset \gamma) = (\alpha \supset \gamma) \wedge_P (\beta \supset \gamma), \tag{76}$$
$$(\alpha \supset \beta \vee_L \gamma) = (\alpha \supset \beta) \vee_P (\alpha \supset \gamma), \tag{77}$$
$$(\alpha \wedge_L \beta \supset \gamma) = (\alpha \supset \gamma) \vee_P (\beta \supset \gamma), \tag{78}$$
$$\xi \& (\beta \vee_L \gamma) = (\xi \& \beta) \vee_L (\xi \& \gamma), \tag{79}$$
$$(\xi \vee_P \tau) \& \gamma = (\xi \& \gamma) \vee_L (\tau \& \gamma). \tag{80}$$
$$\xi \& (\beta \wedge_L \gamma) = (\xi \& \beta) \wedge_L (\xi \& \gamma), \tag{81}$$
$$(\xi \wedge_P \tau) \& \gamma = (\xi \& \gamma) \wedge_L (\tau \& \gamma), \tag{82}$$
$$(\xi \to \tau \vee_P \zeta) = (\xi \to \tau) \vee_P (\xi \to \zeta), \tag{83}$$
$$(\xi \wedge_P \tau \to \zeta) = (\xi \to \zeta) \vee_P (\tau \to \zeta) \quad \textit{[6]}. \tag{84}$$
$$\zeta \otimes (\tau \wedge_P \xi) = (\zeta \otimes \tau) \wedge_P (\zeta \otimes \xi) \quad \textit{[6,9]}. \tag{85}$$

The definition of subdirect products of algebraic systems can be looked up in Hájek's book [8]. In prelinear tied algebras, two comparators are required to be prelinear. This stipulation is needed to establish the following representation:

Theorem 10 (Representation Theorem for Tied Algebras) *[15] A tied algebra is prelinear if and only if it is a subdirect product of a system of tied chains.*

3. Generalized modus ponens with multiple rules

3.1. Zadeh's generalized modus ponens

Let U, W be universes, and let M, M^{pre} be P-valued possibility distributions (shortly, P-possibility distributions) on U. We handle them simply as modal P-fuzzy subsets of U (members of P^U that attain the value 1). Likewise, $Q, Q^{ifr} \in L^W$ are L-possibility distributions on W, which need not be modal. Each of the two symbols X, Z denotes an unknown individual in the universe U or W, respectively. According to Zadeh [22], a *Generalized Modus Ponens* (**GMP**) inference scheme takes the form:

Inference Scheme (I):

Rule:	If X is M then Z is Q
Premise (observation):	X is M^{pre}
Inference:	Z is Q^{ifr}.

Let n be an integer, and denote the ordinal $\{1, \cdots, n\}$ by N. A generalized modus ponens inference scheme with multiple rules takes the form:

Inference Scheme (II):

Rules: If X is M_j then Z is Q_j, $j \in N$

Premise: X is M^{pre}

Inference: Z is Q^{ifr}.

We refer the reader to the comprehensive survey of fundamental contributions to **GMP**, up to 1990, by Dubois and Prade in [5, Section 4], with extensive citations. An *interpretation* of this inference scheme is a concrete mathematical formula to compute Q^{ifr} in terms of M, Q and the observation M^{pre}, subject to a reasonable array of intuitive criteria [5]. These include $Q^{ifr} \geq Q$, isotonicity, the limit case of null specificity ($Q^{ifr} = 1_W$ when $M^{pre} = 1_U$), and the following criterion for the intuitive soundness of interpretations, due to Fukami et al. [7], Turksen and Tian [21] (who call it *the basic requirement for fuzzy reasoning*), and Perfilieva and Lehmke [19,20] (whereby it is called *correctness*):

$$\text{For each } j \in N: \quad Q^{ifr} = Q_j \quad \text{whenever} \quad M^{pre} = M_j. \tag{86}$$

3.2. Interpretations based on Compositional Rule of Inference

Zadeh's *Compositional Rule of Inference* (**CRI**) [22] is the best known type of interpretations of Inference Scheme I of **GMP**. Morsi [12] used in **CRI** the more general connectives of adjointness algebras. Let $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset)$ be an adjointness algebra. According to the combination-projection principle of Zadeh, we take:

$$Q^{ifr}(w) = \sup_{u \in U} M^{pre}(u) \& F(u, w), \qquad w \in W \tag{87}$$

In this section, F denotes the L-possibility distribution on $U \times W$ of the rule of the scheme, which we equate to the L-fuzzy relation $F \in L^{U \times W}$ given by

$$F(u, w) = (M(u) \Rightarrow Q(w)) \in L, \qquad (u, w) \in U \times W. \tag{88}$$

This is not the only possible reading of the rule of the scheme (see Section 4).

Hence, **CRI** yields the following computation of Q^{ifr}:

$$Q^{ifr}(w) = \sup_{u \in U} (M^{pre}(u) \& (M(u) \Rightarrow Q(w))) \in L, \qquad w \in W. \tag{89}$$

The use of a possibly noncommutative conjunction $\& : P \times L \longrightarrow L$ is justified here, because the intuitive meaning of the P-possibility values in the observation M^{pre} may differ from that of the L-values of the rule F. This formulation generalizes that of Dubois and Prade [4], who used a residuated pair $\otimes, \rightarrow$ on $[0, 1]$, whereas Zadeh's original formulation uses only $\& = \min$ on $[0, 1]$.

In the literature, we meet two, intuitively sound, adaptations of Zadeh's Compositional Rule of Inference (**CRI**) to multiple-rules inference schemata. The first approach, called by Turksen and Tian [21] *First Aggregate Then Infer* (FATI), aggregates the given n rules of Inference Scheme II, using infimum, into a single rule modeled by the L-possibility distribution

$$\inf_{j \in N} (M_j(u) \Rightarrow Q_j(w)) \in L, \qquad (u, w) \in U \times W, \tag{90}$$

on the Cartesian product $U \times W$. The conclusion Q^{ifr} (which we also denote by $CRI^{*}_{II}(M^{pre})$) is the L-possibility distribution on W obtained, according to the combination-projection principle, as follows: For all $w \in W$:

$$CRI^{*}_{II}(M^{pre})(w) = \sup_{u \in U} \left(M^{pre}(u) \,\&\, \inf_{j \in N} (M_j(u) \Rightarrow Q_j(w)) \right) \in L. \tag{91}$$

The second approach, called *First Infer Then Aggregate* (FITA), infers a conclusion from each rule separately, and then aggregates those n conclusions using infimum. The inference Q^{ifr}, which we now denote by $CRI_{II}(M^{pre})$, becomes: For all $w \in W$,

$$CRI_{II}(M^{pre})(w) = \inf_{j \in N} \sup_{u \in U} (M^{pre}(u) \,\&\, (M_j(u) \Rightarrow Q_j(w))) \in L. \tag{92}$$

Conclusions obtained through the FATI approach are generally more specific than those obtained through the FITA one; that is, $CRI^{*}_{II}(M^{pre}) \leq_L CRI_{II}(M^{pre})$ (as can be easily seen by comparing (91) with (92)). Moreover, the FATI approach is intuitively more convincing, and it is known to preserve finite joins, while the FITA approach fails to do, even in the classical (binary) case (Dubois and Prade [5]).

Lemma 11 *Under the two approaches FATI and FITA, the* ***CRI****-interpretations of Inference Scheme II satisfy, for all* $j \in N$:

$$CRI^{*}_{II}(M_j) \leq_L CRI_{II}(M_j) \leq_L Q_j. \tag{93}$$

Proof. For every $w \in W$ we have:
$CRI^{*}_{II}(M_j)(w) \leq_L CRI_{II}(M_j)(w) = \inf_{k \in N} \sup_{u \in U} (M_j(u) \,\&\, (M_k(u) \Rightarrow Q_k(w)))$
$\leq_L \sup_{u \in U} (M_j(u) \,\&\, (M_j(u) \Rightarrow Q_j(w))) \leq_L Q_j(w)$ by (11).
This proves (93). ■

Criterion (86) is satisfied by **CRI** (of the type specified above) in all single rule inference schemata (Morsi [12]), but it may fail for multiple rules within both the FATI and FITA approaches. Our purpose, however, is to establish the equivalence of Inference Scheme II to the following inference scheme (that is, the two schemata produce the same conclusions), which satisfies this criterion: For each $j \in N$, let $\overline{Q}_j$ be the conclusion of Inference Scheme II when $M^{pre} = M_j$. We compose:

Inference Scheme (III):

Rules: If X is M_j then Z is $\overline{Q}_j$, $j \in N$

Premise: X is M^{pre}

Inference: Z is $\overline{Q}^{ifr}$.

After we prove the equivalence of Inference Schemata II and III, the latter one will satisfy the basic requirement for fuzzy reasoning automatically, because when $M^{pre} = M_j$, we have $\overline{Q}^{ifr} = Q^{ifr} = \overline{Q}_j$, by the definition of $\overline{Q}_j$. Morsi and Fahmy [14] have already established this equivalence within the special setting of residuated chains on real numbers.

We here extend it to the settings of the more general logical connectives (on lattices P and L); of an adjointness algebra, under the FATI approach (Theorem 12), and of a prelinear tied algebra, under the FITA approach (Theorem 14).

Theorem 12 *In Inference Scheme III, take $\overline{Q}_j = CRI^*_{II}(M_j)$ for all $j \in N$. Then under the FATI approach to **CRI**, and using the connectives of an adjointness algebra $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset)$, the two schemata II and III are equivalent. Moreover, Inference Scheme III satisfies the basic requirement for fuzzy reasoning.*

Proof. In the described case, we denote the conclusions of Inference Scheme III by $CRI^*_{III}(M^{pre})$. For every $j \in N$ we have: $\overline{Q}_j = CRI^*_{II}(M_j) \leq_L Q_j$, by Lemma 11. From this, we deduce directly

$$CRI^*_{III}(M^{pre}) \leq_L CRI^*_{II}(M^{pre}), \tag{94}$$

for all premises M^{pre}. The opposite inequality is derived as follows: Choose and fix $w \in W$. Then

$$\begin{aligned}
CRI^*_{III}(M^{pre})(w) &= \sup_{u\in U}\left(M^{pre}(u) \,\&\, \inf_{j\in N}\left(M_j(u) \Rightarrow \overline{Q}_j(w)\right)\right) \\
&= \sup_{u\in U}\left(M^{pre}(u) \,\&\, \inf_{j\in N}\left(M_j(u) \Rightarrow \sup_{t\in U}\left(M_j(t) \,\&\, \inf_{k\in N}\left(M_k(t) \Rightarrow Q_k(w)\right)\right)\right)\right) \\
&\geq_L \sup_{u\in U}\left(M^{pre}(u) \,\&\, \inf_{j\in N}\left(M_j(u) \Rightarrow \left(M_j(u) \,\&\, \inf_{k\in N}\left(M_k(u) \Rightarrow Q_k(w)\right)\right)\right)\right) \\
&\geq_L \sup_{u\in U}\left(M^{pre}(u) \,\&\, \inf_{k\in N}\left(M_k(u) \Rightarrow Q_k(w)\right)\right) \quad \text{(by (12))} \\
&= CRI^*_{II}(M^{pre})(w).
\end{aligned}$$

This yields the inequality opposite to (94). Hence, equality holds, and the two schemata are equivalent. Consequently, Inference Scheme III satisfies the basic requirement for fuzzy reasoning, as we explained earlier. ■

In contrast with the preceding proof, our proof in the FITA approach needs identity (96) below. This identity fails in a general adjointness algebra, but we now establish its validity in prelinear tied algebras:

Lemma 13 *In a tied algebra $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset, \otimes, \rightarrow)$, if the comparator $\supset$ is prelinear (that is, it satisfies (52)), then*

$$\forall z_1, \cdots, z_n \in L: \quad \sup_{j\in N}\left(z_j \supset \inf_{k\in N} z_k\right) = 1. \tag{95}$$

Also, for all $B, C \in P^U$ and for all $z_1, \cdots, z_n \in L$:

$$\sup_{u\in U}\left(B(u) \,\&\, \left(C(u) \Rightarrow \inf_{k\in N} z_k\right)\right) = \inf_{k\in N}\sup_{u\in U}\left(B(u) \,\&\, (C(u) \Rightarrow z_k)\right). \tag{96}$$

Proof. We have by (78) then (6):

$$\sup_{j\in N}\left(z_j \supset \inf_{k\in N} z_k\right) = \left(\inf_{j\in N} z_j \supset \inf_{k\in N} z_k\right) = 1.$$

This proves (95). We also have

$$\left(\inf_{j\in N}\sup_{u\in U}\left(B(u)\,\&\,(C(u)\Rightarrow z_j)\right)\right) \supset \left(\sup_{t\in U}\left(B(t)\,\&\,\left(C(t)\Rightarrow \inf_{k\in N} z_k\right)\right)\right)$$

$$= \sup_{j\in N}\inf_{u\in U}\left(\left(B(u)\,\&\,(C(u)\Rightarrow z_j)\right)\supset\left(\sup_{t\in U}\left(B(t)\,\&\,\left(C(t)\Rightarrow\inf_{k\in N} z_k\right)\right)\right)\right)$$

(by (78), (3))

$$\geq_P \sup_{j\in N}\inf_{u\in U}\left(\left(B(u)\,\&\,(C(u)\Rightarrow z_j)\right)\supset\left(B(u)\,\&\,\left(C(u)\Rightarrow\inf_{k\in N} z_k\right)\right)\right)$$

$$\geq_P \sup_{j\in N}\inf_{u\in U}\left(\left(C(u)\Rightarrow z_j\right)\supset\left(C(u)\Rightarrow\inf_{k\in N} z_k\right)\right) \quad \text{(by (31))}$$

$$\geq_P \sup_{j\in N}\left(z_j\supset\inf_{k\in N} z_k\right) \quad \text{(by (30))}$$

$= 1$, by (95).

This yields, through the comparator property (6):

$$\inf_{j\in N}\sup_{u\in U}\left(B(u)\,\&\,(C(u)\Rightarrow z_j)\right)\leq_L \sup_{t\in U}\left(B(t)\,\&\,\left(C(t)\Rightarrow\inf_{k\in N} z_k\right)\right).$$

The opposite inequality follows directly from the fact that $\Rightarrow, \&$ are isotone in the right argument. Thus, identity (96) holds. ■

Theorem 14 *In Inference Scheme III, take $\overline{Q}_j = CRI_{II}(M_j)$ for all $j\in N$. Then under the FITA approach to* ***CRI****, and using the connectives of a tied algebra Λ in which the comparator $\supset$ is prelinear, the two inference schemata II and III are equivalent. Moreover, Inference Scheme III satisfies the basic requirement for fuzzy reasoning.*

Proof. In the described case, we denote the conclusion of Inference Scheme III by $CRI_{III}(M^{pre})$. From (93) we conclude that $\overline{Q}_j \leq_L Q_j$ for all $j\in N$. Hence, we have for all premises M^{pre}: $CRI_{III}(M^{pre})\leq_L CRI_{II}(M^{pre})$.

We prove the opposite inequality. Choose and fix $w\in W$:

$$CRI_{III}(M^{pre})(w) = \inf_{j\in N}\sup_{u\in U}\left(M^{pre}(u)\,\&\,\left(M_j(u)\Rightarrow\overline{Q}_j(w)\right)\right)$$

$$= \inf_{j\in N}\sup_{u\in U}\left(M^{pre}(u)\,\&\,\left(M_j(u)\Rightarrow\inf_{k\in N}\sup_{t\in U}\left(M_j(t)\,\&\,(M_k(t)\Rightarrow Q_k(w))\right)\right)\right)$$

$$= \inf_{j\in N}\inf_{k\in N}\sup_{u\in U}\left(M^{pre}(u)\,\&\,\left(M_j(u)\Rightarrow\sup_{t\in U}\left(M_j(t)\,\&\,(M_k(t)\Rightarrow Q_k(w))\right)\right)\right)$$

(by (96))

$$\geq_L \inf_{j\in N}\inf_{k\in N}\sup_{u\in U}\left(M^{pre}(u)\,\&\,\left(M_j(u)\Rightarrow M_j(u)\,\&\,(M_k(u)\Rightarrow Q_k(w))\right)\right)$$

$$\geq_L \inf_{k\in N}\sup_{u\in U}\left(M^{pre}(u)\,\&\,(M_k(u)\Rightarrow Q_k(w))\right) \quad \text{(by (12))}$$

$= CRI_{II}(M^{pre})(w)$.

This completes the proof of the equivalence of schemata II and III, from which we conclude that Scheme III satisfies the basic requirement for fuzzy reasoning. ■

In the case of **CRI** on a (prelinear) tied algebras, Theorems 12 and 14 indicate that Criterion (86) is unnecessarily restrictive (cf. Morsi and Fahmy [14]): The authors believe that the violation of this criterion, by an inference system using **CRI**, merely signifies that the scheme's If-Then rules are not given, individually, in their fullest informative power.

But, those shortcomings of individual rules are compensated for by the other rules. In this sense, Inference Scheme III will be a better rendering of the given If-Then rules than Inference Scheme II.

3.3. Interpretations based on Consequent Dilation Rule

In the following, $\subseteq_P$ stands for the P-fuzzy relation on P^U given by

$$B_1 \subseteq_P B_2 = \inf_{u \in U} (B_1(u) \to B_2(u)) \in P, \tag{97}$$

where B_1, B_2 are two P-fuzzy subsets of U. It is the measure of subsethood of B_1 in B_2 (cf [2]). Likewise, $\subseteq_L$ stands for the P-fuzzy relation on L^W given by

$$C_1 \subseteq_L C_2 = \inf_{w \in W} (C_1(w) \supset C_2(w)) \in P,$$

where C_1, C_2 are two L-fuzzy subsets of W, and it is the measure of subsethood of C_1 in C_2.

The two P-fuzzy relations $\subseteq_P$ and $\subseteq_L$ inherit many of the properties of the R-implication $\to$ and the comparator $\supset$, respectively. In particular, $C_1 \subseteq_L C_2$ will be 1_P if and only if $C_1 \leq_L C_2$ (by the comparator property (6)). Likewise, $B_1 \subseteq_P B_2$ will be 1_P if and only if $B_1 \leq_P B_2$. Also, these two fuzzy relations are antitone in the first argument, isotone in the second argument, and both are $\otimes$-transitive, as the following lemma asserts:

Lemma 15 *The following inequalities hold in a tied algebra Λ: For all $A, B, C \in P^U$ and for all $D, E, Q \in L^W$:*

$$(D \subseteq_L E) \leq_P [(Q \subseteq_L D) \to (Q \subseteq_L E)], \tag{98}$$
$$(Q \subseteq_L D) \leq_P [(D \subseteq_L E) \to (Q \subseteq_L E)], \tag{99}$$
$$(Q \subseteq_L D) \otimes (D \subseteq_L E) \leq_P (Q \subseteq_L E), \tag{100}$$
$$(B \subseteq_P C) \leq_P [(A \subseteq_P B) \to (A \subseteq_P C)], \tag{101}$$
$$(A \subseteq_P B) \leq_P [(B \subseteq_P C) \to (A \subseteq_P C)], \tag{102}$$
$$(A \subseteq_P B) \otimes (B \subseteq_P C) \leq_P (A \subseteq_P C). \tag{103}$$

Proof. We have by (32),

$(D \subseteq_L E) \leq_P \inf_{w \in W} [(Q(w) \supset D(w)) \to (Q(w) \supset E(w))]$
$\leq_P [\inf_{w \in W} (Q(w) \supset D(w)) \to \inf_{w \in W} (Q(w) \supset E(w))]$ (by (16)).

This proves (98). Inequalities (99) and (100) result by applying Residuation to (98). Inequalities (101), (102) and (103) have a similar proof. ■

An alternative interpretation of Inference Scheme II of **GMP** has been suggested by Magrez and Smets [10], and studied in [14] under the name "Consequent Dilation Rule (**CDR**)". In this section, we use in **CDR** the more general connectives of tied algebras.

Let $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset, \otimes, \to)$ be a tied algebra. Then according to **CDR**, the conclusion Q^{ifr} (which we also denote by $CDR_{II}(M^{pre})$) is the L-possibility distribution on W obtained as follows: For all $w \in W$:

$$CDR_{II}(M^{pre})(w) = \inf_{j \in N} (\inf_{u \in U} (M^{pre}(u) \to M_j(u)) \Rightarrow Q_j(w)) \in L, \tag{104}$$

or, briefly,

$$CDR_{II}\left(M^{pre}\right)(w) = \inf_{j\in N}((M^{pre} \subseteq_P M_j) \Rightarrow Q_j(w)). \tag{105}$$

This formulation generalizes that of Morsi and Fahmy [14], who considered only residuated algebras on $[0,1]$.

CDR satisfies a number of intuitive demands on fuzzy reasoning. These include that the inference of rule number j is greater or equal to Q_j (hence, the name of the method), and it will be Q_j if $M^{pre} \leq_L M_j$. Also, the conclusion $CDR_{II}\left(M^{pre}\right)$ is monotone in $M^{pre}, Q_1, \cdots, Q_n$, and antimonotone in $M_1, \cdots, M_n$. In the binary case, **CDR** agrees with reasoning within the FITA approach, because when all fuzzy sets under consideration are crisp, $CDR_{II}\left(M^{pre}\right)$ becomes W (nothing to infer) if $M^{pre} \nsubseteq M_j$.

CDR is computationally faster than **CRI**. This is because, for every observation M^{pre}, the n values of subsethood $M^{pre} \subseteq_P M_j$ are computed once, then they are used at all elements w of W. The saving of computer time in **CDR**, over **CRI**, is at the expense of the specificity of inference results (cf. [14]). That is, **CRI** (92) produces more specific inference results than **CDR** (104) does.

Theorem 16 *For all* $M^{pre} \in P^U$, $CDR_{II}\left(M^{pre}\right) \geq_L CRI_{II}(M^{pre})$.

Proof. We have for all $M^{pre} \in P^U$ and for all $w \in W$,

$CRI_{II}\left(M^{pre}\right)(w) = \inf_{k\in N} \sup_{u\in U}\left(M^{pre}(u) \,\&\, (M_j(u) \Rightarrow Q_j(w))\right)$
$\leq_L \inf_{k\in N} \sup_{u\in U}((M^{pre}(u) \rightarrow M_j(u)) \Rightarrow Q_j(w))$ (by (42))
$\leq_L \inf_{j\in N}(\inf_{u\in U}(M^{pre}(u) \rightarrow M_j(u)) \Rightarrow Q_j(w))$ ($\Rightarrow$ is antitone in the left argument)
$= \inf_{j\in N}((M^{pre} \subseteq_P M_j) \Rightarrow Q_j(w))$
$= CDR_{II}\left(M^{pre}\right)(w)$. ■

We have the following limitations on the dilation of conclusions under **CDR**, and under **CRI** in both the FITA (92) and FATI (91) approaches:

Theorem 17 *For all* $j \in N$ *and all* $M = M^{pre} \in P^U$:

$$(CRI^*_{II}(M) \subseteq_L Q_j) \geq (CRI_{II}(M) \subseteq_L Q_j) \geq (CDR_{II}(M) \subseteq_L Q_j) \geq (M \subseteq_P M_j). \tag{106}$$

Consequently,

$$CDR_{II}(M_j) \leq_L Q_j. \tag{107}$$

Proof. By (10), we have for all $j \in N$ and all $w \in W$:

$(M^{pre} \subseteq_P M_j) \leq_P (((M^{pre} \subseteq_P M_j) \Rightarrow Q_j(w)) \supset Q_j(w))$
$\leq_P (\inf_{k\in N}((M^{pre} \subseteq_P M_k) \Rightarrow Q_k(w)) \supset Q_j(w))$ (by (35))
$= (CDR_{II}\left(M^{pre}\right)(w) \supset Q_j(w))$
$\leq_P (CDR_{II}\left(M^{pre}\right) \subseteq_L Q_j)$.

Since $CRI^*_{II}\left(M^{pre}\right) \leq_L CRI_{II}\left(M^{pre}\right) \leq_L CDR_{II}\left(M^{pre}\right)$ (Theorem 16), we obtain the other two inequalities in (106) by (99). Inequality (107) is an obvious consequence of (106). ■

The obvious reading is that the degree of subsethood of M^{pre} in M_j is the lower bound of the degree of subsethood of the conclusion $CDR_{II}(M^{pre})$ in Q_j. In particular, $CDR_{II}(M^{pre}) \leq_L Q_j$ if $M^{pre} \leq_P M_j$, and in the case of a single rule, $CDR_{II}(M^{pre}) = Q_1$ if $M^{pre} \leq_P M_1$, i.e. **CDR** satisfies Criterion (86) in single-rule inference schemata. But it may fail for multiple rules, even in the binary case. Our purpose, however, is to establish the equivalence of Inference Scheme II to Inference Scheme III, in which the basic requirement for fuzzy reasoning is satisfied by **CDR**[6].

Theorem 18 *In Inference Scheme III, take $\overline{Q}_j = CDR_{II}(M_j)$ for all $j \in N$. Then using the connectives of a tied algebra Λ, the two inference schemata II and III are equivalent as far as **CDR** is concerned. Moreover, Inference Scheme III satisfies the basic requirement for fuzzy reasoning.*

Proof. In the described case, we denote the conclusion of Inference Scheme III by $CDR_{III}(M^{pre})$. From (107) we conclude that $\overline{Q}_j \leq_L Q_j$ for all $j \in N$. Hence, we have for all premises M^{pre}: $CDR_{III}(M^{pre}) \leq_L CDR_{II}(M^{pre})$.

We prove the opposite inequality. For all premises M^{pre} and $w \in W$:

$$CDR_{III}(M^{pre})(w) = \inf_{j\in N}((M^{pre} \subseteq_P M_j) \Rightarrow \overline{Q}_j(w))$$
$$= \inf_{j\in N}((M^{pre} \subseteq_P M_j) \Rightarrow \inf_{i\in N}((M_j \subseteq_P M_i) \Rightarrow Q_i(w)))$$
$$= \inf_{j\in N}\inf_{i\in N}((M^{pre} \subseteq_P M_j) \Rightarrow ((M_j \subseteq_P M_i) \Rightarrow Q_i(w))) \quad \text{(by (3))}$$
$$= \inf_{j\in N}\inf_{i\in N}((M^{pre} \subseteq_P M_j) \otimes (M_j \subseteq_P M_i) \Rightarrow Q_i(w)) \quad \text{(by (21))}$$
$$\geq_L \inf_{i\in N}((M^{pre} \subseteq_P M_i) \Rightarrow Q_i(w)) \quad \text{(by (103))}$$
$$= CDR_{II}(M^{pre}).$$

This completes the proof of the equivalence of schemata II and III, from which we conclude that Scheme III satisfies the basic requirement for fuzzy reasoning, by virtue of the reason expounded in the paragraph preceding Theorem 12. ■

A method of legitimately improving the specificity of inference results under **CDR** is described in the next theorem.

Theorem 19 *In Inference Scheme III, take $\overline{Q}_j = CRI^*_{II}(M_j)$ for all $j \in N$, and denote the conclusion under **CDR** of Inference Scheme III by $CDR^*_{III}(M^{pre})$, using the connectives of a tied algebra Λ. Then Inference Scheme III satisfies the basic requirement for fuzzy reasoning under **CDR**, and it is related to Scheme II as follows: for every observation $M^{pre} \in P^U$:*

$$CRI^*_{II}(M^{pre}) \leq_L CDR^*_{III}(M^{pre}) \leq_L CDR_{II}(M^{pre}). \tag{108}$$

Proof. We have $CRI^*_{II}(M^{pre}) = CRI^*_{III}(M^{pre})$ (by Theorem 12) $\leq_L CDR^*_{III}(M^{pre})$ (by Theorem 16) $\leq_L CDR_{II}(M^{pre})$, because every $\overline{Q}_j$ is less or equal to Q_j (Lemma 11). This proves (108).

In particular, we have for every $j \in N$:

[6] In [14] this equivalence has already been established within the special setting of residuated chains on real numbers. Here we generalize this result to tied algebras on two lattices P and L.

$\overline{Q}_j = CRI^*_{II}(M_j) \leq_L CDR^*_{III}(M_j) \leq_L \overline{Q}_j$, by (107).

The equalities $CDR^*_{III}(M_j) = \overline{Q}_j$ now ensue for all j, establishibg the basic requirement for fuzzy reasoning for the interpretation CDR^*_{III}. ■

The inequalities in (108) ensure that the increase of specificity of inferences by using **CDR** in Inference Scheme III, over those obtained by applying **CDR** to Inference Scheme II, is quite legitimate. Evidently, applications of **CDR** to Inference Schemata III and II require roughly the same operating time. So, the computational advantage of **CDR** is maintained in CDR^*_{III}. However, a small time will have to be spent once in preparing Inference Scheme III.

Example 20 *We denote the opposite lattice of a lattice P by P^{op}. By an* S-type implication *we mean an implication $\Rightarrow: P \times P \longrightarrow P$ that is derived from an object-conjunction $\otimes$ on P, together with a self-inverse lattice isomorphism $\neg : P^{op} \longrightarrow P$, in the following manner:*

$$x \Rightarrow c = \neg (x \otimes (\neg c)), \qquad x, c \in P. \tag{109}$$

The function $\neg$ is understood to interpret negation *on P. Accordingly, S-type implications provide a general framework for a multiple-valued logic in which an entailment "If ξ then γ" is construed as "Not (ξ and (not γ))".*

Let $\Rightarrow$ be an S-type implication, and let $0, 1$ be the bounds of P. Then

$$0 = \neg 1, \tag{110}$$

$$1 = \neg 0, \tag{111}$$

$$\forall x \in P: \quad \neg x = x \Rightarrow 0 = x \supset 0. \tag{112}$$

Also, the object-conjunction $\otimes$ is the unique binary operation on P that ties $\Rightarrow$. It follows that $\Rightarrow$ satisfies all the universal properties of tied algebras, stated in Section 2.3, including the inequalities in Theorem 3. In the tied algebra $\Lambda_S = (P, \leq_P, 1, 0, \Rightarrow, \&, \supset, \otimes, \rightarrow)$, $\rightarrow$ is the residuum of $\otimes$, and the remaining two connectives $\&$ and $\supset$ are given by:

$$x \& b = \neg (x \rightarrow (\neg b)), \tag{113}$$

$$b \supset c = (\neg c) \rightarrow (\neg b). \tag{114}$$

The conclusions of Inference Scheme II, drawn from an observation by means of the methods of Sections 3.2 and 3.3, under the connectives of the algebra Λ_S, are computed as follows: For all $w \in W$:

$$\begin{aligned}
CRI^*_{II}(M^{pre})(w) &= \sup_{u \in U} \left(M^{pre}(u) \,\&\, \inf_{j \in N} (M_j(u) \Rightarrow Q_j(w)) \right) \\
&= \sup_{u \in U} \left(\neg \left(M^{pre}(u) \rightarrow \left(\neg \inf_{j \in N} \neg (M_j(u) \otimes (\neg Q_j(w))) \right) \right) \right) \\
&= \sup_{u \in U} \left(\neg \left(M^{pre}(u) \rightarrow \left(\sup_{j \in N} (M_j(u) \otimes (\neg Q_j(w))) \right) \right) \right) \\
&= \neg \left(\inf_{u \in U} \left(M^{pre}(u) \rightarrow \left(\sup_{j \in N} (M_j \otimes (\neg Q_j(w))) \right)(u) \right) \right) \\
&= \neg \left(M^{pre} \subseteq_P \left(\sup_{j \in N} (M_j \otimes (\neg Q_j(w))) \right) \right),
\end{aligned}$$

and

$$CRI_{II}(M^{pre})(w) = \inf_{j\in N}\sup_{u\in U}(M^{pre}(u)\,\&\,(M_j(u)\Rightarrow Q_j(w))) = \inf_{j\in N}(\neg(M^{pre}\subseteq_P(M_j\otimes(\neg Q_j(w)))))$$

$$= \neg\left(\sup_{j\in N}(M^{pre}\subseteq_P(M_j\otimes(\neg Q_j(w))))\right),$$

$$CDR_{II}(M^{pre})(w) = \inf_{j\in N}((M^{pre}\subseteq_P M_j)\Rightarrow Q_j(w)) = \inf_{j\in N}(\neg((M^{pre}\subseteq_P M_j)\otimes(\neg Q_j(w))))$$

$$= \neg(\sup_{j\in N}((M^{pre}\subseteq_P M_j)\otimes(\neg Q_j(w)))).$$

4. Fuzzy Controllers

In this section, we work again with Inference Schemata I and II of **GMP** (of Section 3.1) with an interpretation based on a particular case of the compositional rule of inference, called Generalized Conjunctive Rule (**GCR** for short), due to Hájek [8]. **GCR** is not a frequently used type of interpretations of Inference Scheme I of **GMP**, but we get it naturally from the compositional rule and we shall use our analysis of this particular case to investigate the principles of fuzzy control (the most broadly used application of fuzzy logic) in general.

Here, we shall consider the **GCR** as particular case of the compositional rule of inference, within the settings of the more general logical connectives (on lattices P and L) of a tied algebra. To this end, we take in (87) $F(u,w)$ to be $M(u)\&Q(w)$ for some $M\in P^U$ and $Q\in L^W$. Thus we get the following variant of (89) which define the inference result Q^{ifr} of Inference Scheme I :

$$Q^{ifr}(w) = \sup_{u\in U}(M^{pre}(u)\,\&\,(M(u)\,\&Q(w)))\in L, \qquad w\in W. \tag{115}$$

This formulation generalizes that in [8], where a residuated pair $\otimes,\rightarrow$ on lattice L is used , whereas Mamdani's original formulation uses only $\& = \min$ on $[0,1]$, see [11].

Theorem 21 *Under **GCR** interpretation of Inference Scheme I , and using the connectives of a tied algebra Λ, we have:*

$$\sup_{u\in U} M^2(u)\otimes(M\subseteq_P M^{pre})\leq_P(Q\subseteq_L Q^{ifr}). \tag{116}$$

Moreover, Inference Scheme I satisfies the basic requirement for fuzzy reasoning for all modal P-fuzzy sets M.

Proof. We have for all $w\in W$,

$$(M\subseteq_P M^{pre})$$
$$\leq_P(\sup_{u\in U}(M(u)\&(M(u)\&Q(w)))\supset\sup_{u\in U}(M^{pre}(u)\&(M(u)\&Q(w))))\text{ (by (31))}$$
$$=(\sup_{u\in U}(M^2(u)\&Q(w))\supset\sup_{u\in U}(M^{pre}(u)\&(M(u)\&Q(w))))\text{ (by (22))}$$
$$=(\sup_{u\in U}(M^2(u)\&Q(w))\supset Q^{ifr}(w))\text{ (by (115))}$$

$= ((\sup_{u\in U} M^2(u) \& Q(w)) \supset Q^{ifr}(w))$ (by (7))
$= (\sup_{u\in U} M^2(u) \to (Q(w) \supset Q^{ifr}(w)))$ (by strong adjointness, (23))
Thus, by Residuation, we get (116).
On the other hand, we have for all $w \in W$,
$Q^{ifr}(w) = \sup_{u\in U}(M^{pre}(u)\&(M(u)\&Q(w)))$
$= \sup_{u\in U}((M^{pre}(u) \otimes M(u))\&Q(w))$ (by tiedness, (22))
$= (\sup_{u\in U}(M^{pre}(u) \otimes M(u))\&Q(w))$ (by (7))
$\leq_L Q(w)$
This proves that $Q^{ifr} \leq_L Q$, from which and by (116) we conclude that Scheme I satisfies the basic requirement for fuzzy reasoning or all modal P-fuzzy sets M. ■

Let us now turn to the multiple-rule Inference Scheme (II) (of Section 3.1) with an interpretation based on **GCR**. Accordingly, the GCR-inference result Q^{ifr} of Scheme (II) (which we also denote by $GCR_{II}(M^{pre})$) becomes: For all $w \in W$:

$$GCR_{II}(M^{pre})(w) = \sup_{j\in N} \sup_{u\in U} (M^{pre}(u)\&(M_j(u)\&Q_j(w))) \in L. \tag{117}$$

The form of **GCR** given in (117), above, belongs to FITA approach. A version **GCR*** of **GCR**, that belongs to FATI approach can be constructed; according to which the inference result Q^{ifr} of Scheme (II) (which we also denote by $GCR^*_{II}(M^{pre})$) will be given by: For all $w \in W$:

$$GCR^*_{II}(M^{pre})(w) = \sup_{u\in U} \left(M^{pre}(u)\&\sup_{j\in N}(M_j(u)\&Q_j(w)) \right) \in L. \tag{118}$$

In fuzzy control (based on **GCR** interpretations), there is a logical mismatch caused by the fact that "fuzzy IF-THEN rules" are presented as implications but then used to construct a fuzzy relation having little to do with any implication, at least at first glance (the relation is defined by a disjunction of conjunctions). Attempts to call e.g. the min conjunction a "Mamdani implication" (see e.g. [3]) must be strictly rejected since minimum as implication does not obey the principle saying that the truth function of a connective must behave classically for extremal values $0, 1$. It has slowly become clear that fuzzy control deals with approximation of functions on the basis of pieces of fuzzy information of the kind "for arguments approximately equal c_i the image is approximately equal to d_i" (see [8]).

Hájek, in [8], based his elaboration of systems of "fuzzy IF-THEN rules" on the notion of a fuzzy function F (using the notion of a similarity) and n examples $F(c_i, d_i)$ (c_i arguments, d_i images). $M_i(u)$ said "u is similar to c_i ", $Q_i(w)$ said "w is similar to d_i ". He showed that $\sup_{j\in N}(M_j(u)\&Q_j(w)) \leq F \leq \inf_{j\in N}(M_j(u) \Rightarrow Q_j(w))$ (with $\otimes = \&$ and $\Rightarrow = \to$ on $L = P$). Also, he showed that $\sup_{i\in N} M_i^2(u)$ (saying that u very much satisfies some M_i) implies $\sup_{j\in N}(M_j(u)\&Q_j(w)) = \inf_{j\in N}(M_j(u) \Rightarrow Q_j(w))$.

In this section we investigate the (logical) principles of fuzzy control in general, without relating it to the notion of similarity.

The most common way to determine the inference result Q^{ifr} of Inference Scheme (II), in fuzzy logic controllers, is referred to as a method of *interpolation* (**IntP**, for short), suggested by Zadeh [22]. In our general setting, **IntP** supplies the following definition for Q^{ifr} (which we also denote by $IntP_{II}(M^{pre})$): For all $w \in W$:

$$IntP_{II}(M^{pre}) = \sup_{j\in N}((M^{pre} \Cap M_j) \& Q_j(w)) \in L. \tag{119}$$

in which

$$M^{pre} \Cap M_j = \sup_{u\in U}(M^{pre}(u) \otimes M_j(u)) \in P.$$

$M^{pre} \Cap M_j$ calculates the degree of consistency between the given fact and the antecedent of each IF-THEN rule j in terms of the height of intersection of the P-fuzzy sets $M^{pre}, M_j \in P^U$.

The following theorem asserts that the three interpretations of Inference Scheme (II) defined by (117), (118) and (119) are equivalent (i.e. compute the same inference result.)

Theorem 22 *Using the connectives of a tied algebra* Λ*, we have:*
$IntP_{II}(M^{pre}) = GCR_{II}(M^{pre}) = GCR^*_{II}(M^{pre})$.

Proof. $IntP_{II}(M^{pre})(w) = \sup_{j\in N}(\sup_{u\in U}(M^{pre}(u) \otimes M_j(u)) \& Q_j(w))$
$= \sup_{j\in N} \sup_{u\in U}((M^{pre}(u) \otimes M_j(u)) \& Q_j(w))$ (by (7))
$= \sup_{j\in N} \sup_{u\in U}(M^{pre}(u) \& (M_j(u) \& Q_j(w)) = GCR_{II}(M^{pre})(w)$ (by (22))
$= \sup_{u\in U} \sup_{j\in N}(M^{pre}(u) \& (M_j(u) \& Q_j(w))$
$= \sup_{u\in U}(M^{pre}(u) \& \sup_{j\in N}(M_j(u) \& Q_j(w)) = GCR^*_{II}(M^{pre})(w)$(by (79)). ■

We denote by **MAMD** (resembling the name Mamdani, see [11]) any one of the three interpretations **GCR**, **GCR***, or **IntP**, since all these interpretations are equivalent (by Theorem 22). Accordingly, we denote by $MAMD_{II}(M^{pre})$ the conclusion Q^{ifr} of Inference Scheme (II), when computed by **GCR** (117), **GCR*** (118) or **IntP** (119).

Although the three methods **GCR**, **GCR*** and **IntP** are equivalent, **IntP** indicates a faster arrangement of computation, whereby, for every observation M^{pre}, the n values of consistency $M^{pre} \Cap M_j$ are computed once, then they are used at all $w \in W$.

Example 23 *The conclusions of Inference Scheme II, drawn from an observation by means of* ***MAMD*** *(119) (equivalently (117) or (118)), under the connectives of the algebra* $\Lambda_S = (P, \leq_P, 1, 0, \Rightarrow, \&, \supset, \otimes, \rightarrow)$ *(see Example 20), are computed as follows: For all* $w \in W$:

$$MAMD_{II}(M^{pre})(w) = \sup_{j\in N}((M^{pre} \Cap M_j) \& Q_j(w))$$
$$= \sup_{j\in N}\neg((M^{pre} \Cap M_j) \rightarrow \neg Q_j(w))$$
$$= \neg \inf_{j\in N}((M^{pre} \Cap M_j) \rightarrow \neg Q_j(w)).$$

GCR (117) (equivalently (118) or (119)) defines a functional associating to each P-fuzzy subset M^{pre} of U the corresponding L-fuzzy subset Q^{ifr} of W (interpreting the inference result). (Note that in fuzzy control this is used to define a crisp mapping of U into W: one first uses a *fuzzification* operation, associating to each $u \in U$ a fuzzy set M^{pre} ("approximately u"), then applies the functional to get Q^{ifr} and finally applies a *defuzzification* procedure converting the fuzzy set Q^{ifr} into a crisp output $w \in W$. We shall not discuss the operations of fuzzification and defuzzification.)

Let DNF and CNF denote the L-possibility distributions on $U \times W$ of the rule of the scheme, which we equate to the L-fuzzy relations $DNF, CNF \in L^{U \times W}$ given by

$$DNF(u,w) = \sup_{j \in N}(M_j(u) \& Q_j(w)) \in L, \qquad (u,w) \in U \times W, \tag{120}$$

$$CNF(u,w) = \inf_{j \in N}(M_j(u) \Rightarrow Q_j(w)) \in L, \qquad (u,w) \in U \times W, \tag{121}$$

In [8] various results on the relation between (120) and (121) has been characterized within the BL-algebras (using the notion of a similarity). Here, we just restrict ourselves to following result relating (120) and (121) in our new settings of the more general logical connectives (on lattices P and L) of a tied algebra.

Theorem 24 *Using the connectives of a tied algebra (on lattices P and L), we have for all $u \in U$ and $w \in W$:*
$\sup\limits_{i \in N} M_i^2(u) \leq_P (CNF(u,w) \supset DNF(u,w))$.

Proof. Choose and fix $w \in W$ and $u \in U$. For every $i \in N$ we have:

$$\begin{aligned} &CNF(u,w) \\ &\quad = \inf_{j \in N}(M_j(u) \Rightarrow Q_j(w)) \text{ (by (121))} \\ &\quad \leq_L (M_i(u) \Rightarrow (M_i(u) \Rightarrow \sup_{j \in N}(M_j(u) \& Q_j(w))) \text{ (by (12) and the monotonicity of } \Rightarrow) \\ &\quad = (M_i(u) \Rightarrow (M_i(u) \Rightarrow DNF(u,w)) \text{ (by (120))} \\ &\quad = (M_i^2(u) \Rightarrow DNF(u,w)) \text{ (by Tiedness (21)).} \end{aligned}$$

Thus, by Adjointness 1, we get the result. ∎

Theorem 24 says that for each $u \in U$ and $w \in W$ the degree in which u satisfies $\sup\limits_{i \in N} M_i^2(u)$ (i.e., by which u *very much* satisfies some M_i) is a lower bound for the degree in which (u,w) satisfies $CNF(u,w) \supset DNF(u,w)$.

In the following, for some $Q_1, Q_2 \in L^W$, the notation $Q_1 \equiv Q_2$ is used to abbreviate the writing of a relation $(Q_1 \subseteq_L Q_2) \otimes (Q_2 \subseteq_L Q_1)$. Also, Let $DNF \subseteq_L CNF$ stand for $\inf\limits_{u \in U} \inf\limits_{w \in W}(DNF(u,w) \supset CNF(u,w))$. Recall that $CRI_{II}^*(M^{pre})$ is the inference result of Scheme (II) computed by (91). The following theorem relates $CRI_{II}^*(M^{pre})$ and $MAMD_{II}(M^{pre})$ as follows:

Theorem 25 *Using the connectives of a tied algebra (on lattices P and L), we have the following:*
(i) $\inf\limits_{u \in U}(\sup\limits_{i \in N} M_i^2(u)) \leq_P (CRI_{II}^*(M^{pre}) \subseteq_L MAMD_{II}(M^{pre}))$,

(ii) $(DNF \subseteq_L CNF) \leq_P (MAMD_{II}(M^{pre}) \subseteq_L CRI^*_{II}(M^{pre}))$,
(iii) $(DNF \subseteq_L CNF) \otimes \inf_{u \in U}(\sup_{i \in N} M_i^2(u)) \leq_P (MAMD_{II}(M^{pre}) \equiv CRI^*_{II}(M^{pre}))$.

Proof. i- By Theorem 24 and monotonicity of & we get for all $w \in W$:

$$\begin{aligned}
&\inf_{u \in U}(\sup_{i \in N} M_i^2(u)) \\
&\quad \leq_P (\sup_{u \in U}(M^{pre}(u) \& CNF(u,w)) \supset \sup_{u \in U}(M^{pre}(u) \& DNF(u,w))) \\
&\quad = (CRI^*_{II}(M^{pre})(w) \supset MAMD_{II}(M^{pre})(w)) \text{ (by (91) and (118))} \\
&\quad \leq_P \inf_{w \in W}(CRI^*_{II}(M^{pre})(w) \supset MAMD_{II}(M^{pre})(w)).
\end{aligned}$$

ii- Direct by the monotonicity of &.
iii- By combining (i) and (ii) by (27), we get the result. ■

Part (iii) of the last theorem says that, if DNF is sufficiently included in CNF and, for each $u \in U$, u (sufficiently) very much satisfies some M_i then $MAMD_{II}(M^{pre})$ is sufficiently close to $CRI^*_{II}(M^{pre})$. In particular, if $DNF \leq_L CNF$ and for each $u \in U$ there is some i such that $M_i(u) = 1$ then, for all observations M^{pre}, $MAMD_{II}(M^{pre}) = CRI^*_{II}(M^{pre})$.

If M_i, Q_i are interpreted by crisp (0, 1 -valued) subsets of the respective domains then $DNF \leq_L CNF$ is equivalent to $M_i \cap M_j = \emptyset$ for all $i \neq j$ (i.e. they are disjoint). Also, the condition $\inf_{u \in U}(\sup_{i \in N} M_i^2(u))$ above means that the whole domain are covered by all M_i, $i = 1, ..., n$.

In the following we investigate the status of Criterion (86) (cf. Section 3.1) in the case of **MAMD** interpretation of Inference Scheme (II). Criterion (86) is satisfied by **MAMD** in all single rule inference schemes (see Theorem 21). But, this is not generally true in multiple-rule inference schemes.

We do not have analogues of Theorems 12, 14 and 18 for the method **MAMD**. However, we now show that Criterion (86) is satisfied by **MAMD** in a special type of inference schemata.

Theorem 26 *Under **MAMD** interpretation of Inference Scheme II , and using the connectives of a tied algebra* $\Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \&, \supset, \otimes, \to)$, *we have for each* $i \in N$:
i) $\sup_{u \in U}(M^{pre}(u) \otimes M_i(u)) \leq_P (Q_i \subseteq_L MAMD_{II}(M^{pre}))$
ii) $(DNF \subseteq_L CNF) \otimes (M^{pre} \subseteq_L M_i) \leq_P (MAMD_{II}(M^{pre}) \subseteq_L Q_i)$
iii) $(DNF \subseteq_L CNF) \otimes \sup_{u \in U}(M_i^2(u)) \leq_P (MAMD_{II}(M^{pre}) \equiv Q_i)$

Proof. For each $w \in W$, we have

$$\begin{aligned}
&\sup_{u \in U}(M^{pre}(u) \otimes M_i(u)) \& Q_i(w) \\
&\quad \leq_L \sup_{i \in N}\left(\sup_{u \in U}(M^{pre}(u) \otimes M_i(u)) \& Q_i(w)\right) \\
&\quad = MAMD_{II}(M^{pre}))(w).
\end{aligned}$$

Thus, by Adjointness 1, 2, we get

$$\begin{aligned}
&\sup_{u \in U}(M^{pre}(u) \otimes M_i(u)) \leq_P (Q_i(w) \supset MAMD_{II}(M^{pre}))(w)) \\
&\leq_P \inf_{w \in W}(Q_i(w) \supset MAMD_{II}(M^{pre}))(w)).
\end{aligned}$$

This proves (i).
ii) $(DNF \subseteq_L CNF) \otimes (M^{pre} \subseteq_L M_i)$
$\leq_P \inf_{w \in W} \inf_{u \in U} (DNF(u,w) \supset CNF(u,w)) \otimes (M^{pre}(u) \to M_i(u))$
$\leq_P \inf_{w \in W} \inf_{u \in U} ((M^{pre}(u) \& DNF(u,w)) \supset (M_i(u) \& CNF(u,w)))$ (by (27))
$\leq_P \inf_{w \in W} \left(\sup_{u \in U} (M^{pre}(u) \& DNF(u,w)) \supset \sup_{u \in U} (M_i(u) \& CNF(u,w)) \right)$ (by (17))
$\leq_P \inf_{w \in W} (MAMD_{II}(M^{pre})(w) \supset Q_i(w))$.
The last inequality follows by (118) and from Theorem 17 (2) by taking $M^{pre} = M_i$. This proves (ii)
iii) Direct by combining (i) and (ii) by (27) and taking $M^{pre} = M_i$. ■

This theorem shows that the height of intersection of the P-fuzzy subsets M^{pre}, M_i of U is the lower bound of the degree of subsethood of Q_i in $MAMD_{II}(M^{pre})$. In particular, if $M^{pre}(u) \otimes M_i(u) = 1$ for some $u \in U$ then Q_i is included in $MAMD_{II}(M^{pre})$. Moreover, if DNF is sufficiently included in CNF and M_i is (sufficiently) non-empty then Q_i is sufficiently close to $MAMD_{II}(M^{pre})$ whenever $M^{pre} = M_i$. That is Criterion (86) is satisfied by **MAMD** in this special type of inference scheme.

5. Conclusion

We demonstrated the usefulness of tied implications through some applications. We used the connectives of a tied adjointness algebra to interpret Generalized Modus Ponens (**GMP**) inference schemata, in the vein of both the Compositional Rule of Inference (**CRI**) and the Consequent Dilation Rule (**CDR**). We showed that a multiple-rule, generalized modus ponens inference scheme is equivalent, as far as **CRI** or **CDR** are concerned, to a system that satisfies the "basic requirement for fuzzy reasoning". We end by investigating the principles of fuzzy control in general with interpretations based on a particular case of the **CRI**, called Generalized Conjunctive Rule (**GCR**). We showed that the "basic requirement for fuzzy reasoning" is satisfied by **GCR** and using the connectives of a tied algebra in all single rule inference schemes. But, in multiple-rule inference schemes, it is satisfied in a special type of inference scheme. We indicated clearly where prelinearity is needed in some (but not all) of our proofs.

Two more applications of tied implications to fuzzy logic are given by the authors in [15].

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